PERCOLATION, FRACTALS AND THE CRITICAL POINT IN A NUCLEAR REACTOR

VASILIY VASILIY RYAZANOV

Institute for Nuclear Research, Kiev, Ukraine
e-mail: vryazan19@gmail.com

Abstract

The trajectories of neutrons in the reactor, the points of their fission of uranium nuclei, the points of neutron absorption, fission chains and chain reactions are considered from the standpoint of fractal geometry and percolation theory. In the study of the stationary critical operating mode of a nuclear reactor, models of Cayley trees and Laplacian fractals are used. This approach allows us to obtain the neutron multiplication equation and an expression for the critical size of the reactor. Models of irreversible growth and various fractal dimensions are also considered as applied to the evolution of neutrons in a reactor. Prospects for the development of the proposed approach to describing reactors, primarily the kinetics and processes of neutron transfer, are indicated.

Keywords: *Fractals, Percolation, Fission chains, Neutrons, Nuclear reactor*

Introduction

Fractal models of various kinds of systems make it possible to discover new features of seemingly well-known phenomena. Many physical systems are fractal and multifractal. Below we consider some possibilities for describing the motion of neutrons in a reactor using the approaches of percolation [1, 2] and fractality [2-4].

Percolation is the phenomenon of flow or non-flow of liquids through porous materials, electricity through a mixture of conductive and non-conducting particles, and other similar processes. Percolation theory finds application in describing a variety of systems and phenomena, including the spread of epidemics and the reliability of computer networks. Percolation is the moment when a system state appears in which there is at least one continuous path through adjacent conducting nodes from one to the opposite edge. The set of elements through which flow occurs is called a percolation cluster.

Fractal is a structure consisting of parts that are in some sense similar to the whole [5]. Fractal properties manifest themselves especially clearly at the very point of the phase transition, in the critical region. The steady-state operation of a nuclear reactor (NR) takes place precisely at the critical point, and the fractal description should be very important for characterizing the neutron behaviour [6].

In [7], the spread of rumors in the percolation model is compared with a chain reaction. The relations of the theory of percolation [1, 2] are also valid in the general theory of phase transitions [8 - 10]. Fractal concepts were used in the study of highly developed turbulence, inhomogeneous star clusters [11], diffusion-limited aggregation, processes of destruction of matter, the structure of blood, etc. The description of the physical properties of systems with a fractal structure led to the development of analytical methods in the fractal concept based on the use of the mathematical apparatus of fractional order integro-differentiation, since the dimension of space becomes fractional. The need to switch to neutron transfer equations in fractional derivatives may be of practical importance for reactor calculations [12], although there is often no sharp distinction between percolation processes and diffusion [2]. It was noted in [13] that transport processes in percolation clusters, fractal trees, and porous systems must be analyzed anew in order to obtain correct transport equations for such systems. In branching fractal structures, “super-slow” transfer processes can occur, when a physical quantity changes more slowly than the first derivative. The index of the fractional derivative with respect to time corresponds to the proportion of channels (branches) open to flow. The dynamics of diffusion is determined by the random nature of particle motion. A diffusing particle can reach any point in the medium. Percolation is associated with a fractal environment. Below the occurrence threshold, the process of particle propagation is limited to a finite region of the medium. During diffusion from a source, a diffusion front appears that has a fractal structure. In [5], the term shell of a percolation cluster is introduced. Below, from a fractal perspective, we consider the chain reaction processes in the reactor: neutron trajectories, fission of uranium nuclei by neutrons, the formation of new neutrons, their absorption, etc.

Fractal behaviour of fission chains. Percolation models

The model of Cayley trees [14 - 16] or Bethe lattice is applicable to fission chains in NR. A Cayley tree, also called a Bethe lattice, is constructed starting from a central node from which z branches of unit length emanate, forming the first shell of the Cayley tree. The end of each branch is also a node. From each node z - 1 new branches emanate, forming z (z - 1) nodes of the second shell. The process continues ad infinitum. This produces an infinite Cayley tree with z branches emanating from each node.

There are no loops in the system because any two nodes are connected by only one path. In this case, the random nature of branching should be taken into account. You can also apply the theory of random graphs [17]. In [18], objects are considered very close to our study subject - random trees (random aggregates without loops). A general relationship between the diffusion index and tree fractal dimension is obtained. Knowledge of the internal properties of clusters allows one to study their dynamic properties. Fractal phenomena can be classified as stochastic phenomena, since there is a close connection between fractal phenomena and statistical distributions [19].

The trajectories of neutron motions in nuclear reactors form a characteristic branching structure of the process for the total number of neutrons. The figure shows examples of trajectories of one neutron introduced into a breeding medium, taking into account those evolutionary events (nuclear fission and neutron absorption) that lead to a change in the size of the neutron population.

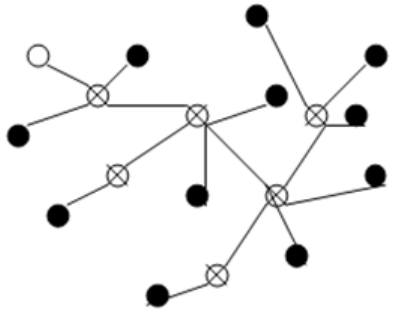


Fig. 1. Trajectories of neutrons and their descendants in the breeding medium:
○ - point at which the initial neutron begins to move;
⊗ - points of fission of nuclei by neutrons;
● - neutron absorption points.

The processes represented by trees are associated with branching random processes [20]. We consider an aspect of the problem determined by the size and nature of the behavior of clusters - nodes connected to each other. By a node we mean a fissile nucleus (or a neutron introduced into the system - the root of a tree [17]), and by a connection - neutron trajectories. Neutron absorption points form so-called hanging ends [17] (vertices of degree 1) or free ends.

At high subcriticality and large negative reactivity values $\rho=(k_{ef}-1)/k_{ef}$ (the effective multiplication factor k_{ef} is much less than unity), the system contains small-sized clusters with a predominant number of hanging ends. If the intensity of neutron death during the time $\Delta t \to 0$ (absorption by the environment or leaving the system) is denoted $\lambda_c\Delta t+0(\Delta t)$, and the intensity of nuclear fission by a neutron $\lambda_f\Delta t+0(\Delta t)$ ($\lambda_f=v\Sigma_f$, where v is the neutron speed; Σ_f is macroscopic fission cross section), then the probability of nuclear fission by a neutron is $c=p=\lambda_f / (\lambda_f + \lambda_c)$. Effective neutron multiplication factor $k_{ef} = p\overline{\nu}$, where $\overline{\nu}$ is the mathematical expectation of the number of secondary neutrons in one fission event. As p increases, the cluster sizes increase. At $p = 1$, all fuel nuclei in the nuclear reactor are separated, and $k_{ef\,max} = \overline{\nu}$ (under such conditions an explosion occurs). For $1 - p << 1$ there is an infinite cluster in the system.

There must be a critical value p_c at which a transition from one mode to another occurs - an infinite cluster appears for the first time. This corresponds to the case $k_{ef} = 1$, $p_c = 1/\overline{\nu}$. The same result in the percolation model was obtained strictly mathematically [1 - 4, 7]. The formation of an infinite cluster represents a phase transition, the beginning of a self-sustaining chain reaction, the critical point of the system. An important role in the theory of phase transitions is played by the concept of an order parameter, that physical quantity that occupies a key place in the processes leading to the transformation. In the theory of percolation clusters, the order parameter is the power of an infinite cluster - the probability that a lattice node will belong to an infinite cluster. The critical behavior of this quantity at $p \to pc$, $p > pc$ is determined by the dependence

$$P\infty = (p - pc)^{\beta}, \qquad (1)$$

where β is one of the so-called critical indicators (scaling indices - in terms of percolation theory) [1, 4]. The value β determines the critical behavior of the power of an infinite cluster $P\infty$. In percolation theory, probability (1) is also called the percolation probability. It serves as the main characteristic of the percolation system. The percolation probability can be used to express properties of physical systems that depend on the topology of large clusters, such as spontaneous magnetization or conductivity. Such quantities as the average number of nodes of the final cluster, the correlation length ξ, the characteristic spatial scale of the cluster at $p < pc$, and at $p > pc$- the characteristic size of the voids in it are also determined.

The value of $P\infty$ in a nuclear reactor is proportional to the deviation $n - nc$, where n is the neutron density, n_c is the critical value of n. Let us denote $\tau = (p-p_c) / p_c = k_{ef} - 1$. The average number of nodes of a finite cluster s (this value is similar to the generalized susceptibility in statistical physics and the fluctuation theory of phase transitions [8, 9]) at $\tau \to 0$ behaves as

$$s \sim |\tau|^{-\gamma}, \quad \tau \ll 1 \qquad (2)$$

where γ is the critical index for s. The correlation length ξ, the characteristic spatial scale of the cluster at $p<p_c$, and at $p>p_c$ the characteristic size of voids in it, near the critical point the average size of the critical cluster at $\tau \to 0$ behaves as

$$\xi \sim |\tau|^{-\nu} \qquad (3)$$

where v is the critical indicator of the value ξ (relative to τ). For nuclear reactors, the influence of external influences (sources, control rods, feedback, system' size, its boundaries, etc.) is significant. If we denote the external field h (set in some effective way depending on the type of influence), then at $h \neq 0$ we can introduce another critical indicator δ

$$P_{\infty}(\tau = 0, h) = h^{1/\delta} \qquad (4)$$

Since $P\infty \sim n - nc$, then external fields, which are sources and sinks of neutrons in nuclear reactors (their physical nature is due to the influence of feedback, control effects, the influence of boundaries, delayed neutrons, etc.), shift the critical point, playing the role of reactivity additives.

The reactivity dependence on such additives is generally nonlinear, and changes in reactivity are not additive.

The value of $pc = 1/\bar{\nu}$ shifts and fluctuations $\bar{\nu}$ - the number of neutrons emitted during the decay of the nucleus. The analogue of thermodynamic heat capacity c_T behaves at $\tau \to 0$ as

$$c_T \sim |\tau|^{-\alpha} \quad (5)$$

where α is the corresponding critical indicator. Formulas (2) and (3) are known in the theory of nuclear reactors, although they were obtained there in a different way.

Thus, Eq. (2) is the neutron multiplication equation:

$$N = (1 - k\,э\,ф)^{-1} \quad (6)$$

those, critical index $\gamma = 1$, Eq. (3) - critical size equation

$$R э ф = \pi M (k\text{ef} - 1)^{-1/2} \quad (7)$$

where R_{ef} is the effective size, a geometric parameter, M is the neutron migration length, indicator $\nu = ½$ (in this case).

Eq. (4) reflects the influence of factors such as control rods and boric acid concentration on the value of the order parameter at the critical point. It is not easy to give an unambiguous interpretation of Eq. (5). But the use of percolation theory and the constructions of fractal theory make it possible to write down a number of other relationships and consider, for example, dynamic critical indices, the dimension of the cluster skeleton, the spectral (fracton) dimension, etc.

Power dependences, as in Eq. (1)-(3), are generally characteristic of random variables with infinitely divisible distributions [19] and their subclass - stable distributions, the connection of which with fractal behavior has been strictly proven for the Wiener process, but which, apparently, are also associated with other random processes corresponding to fractal phenomena.

Critical exponents are related to each other by scaling relations

$$\begin{aligned} &\alpha = 2 - d\nu = 2 - 2\beta - \gamma, \quad \delta = \gamma\beta + 1; \\ &\eta = 2 - \gamma/\nu; \quad d\nu = 2\beta + \gamma \end{aligned} \quad (8)$$

where d is the dimension of space, index η determines the behavior of the correlation function $g(r)$ at large r, when $g(r) \sim r^{-d+2-\eta}$. The classical theory (its presentation for the case of polymer trees is given, for example, in [10]) allows us to obtain indices equal to $\gamma = 1$, $\beta = 1$, $\nu = 1/2$, $\delta = 2$, $\eta = 0$, $\alpha = 1$. The validity of the classical indices for the reactor is explained by the fact that neutrons do not interact and the theory of a self-consistent field is valid. Critical exponents are universal, they do not depend on many rough properties of the chosen model, but are sensitive to various symmetry properties, the presence of long-range correlations, etc., characterizing a whole class of physical phenomena (in the asymptotic limit, near the transition point, where only the maximum term of the expansion is considered, e.g. by τ in Eqs. (2) and (3)).

The characteristic largest size of the finite clusters is determined by the exponent Δ, for which the hyperscaling relation $\Delta = d\nu - \beta$ is valid. At $\tau \neq 0$

$$n_s(\tau) = f(s)\exp(-As|\tau|^{\Delta}) \quad (9)$$

where n_s is the fraction of fission chains, clusters consisting of s neutrons; $f(s)$ is some function that grows no faster than the power s. Clusters with $s_c \sim |\tau|^{-\Delta}$ are called critical. The probability that clusters will have larger sizes decreases exponentially. The characteristic spatial scale of the critical cluster is equal to the correlation length.

The behavior of the system is determined by the relationship between two spatial scales: the minimum length a_0 (a value on the order of the neutron free path λ) and the correlation length ξ. At $\xi >> a_0$ there is a region of intermediate asymptotis $a_0 << r << \xi$. In this region, all characteristics of clusters (measurements on a scale $< \xi$) are similar to their characteristics at the most critical point, when $\tau = 0$, $\xi = \infty$. Their properties in this area are characterized by self-similarity (scale invariance). The reason for the similarity of critical phenomena is the similarity (self-similarity) of geometric objects. A characteristic of such self-similarity is the fractal dimension. At scales smaller than a_0, there is no self-similarity. There are various ways to determine the fractal dimension. For example, in the relations from [5, 16], the total number of particles in the system N is related to the linear size of the system r by the relation

$$N \sim r^{D} \quad (10)$$

where D is the self-similarity dimension, the fractal critical dimension. In [16], a relation was written for the length of a broken curve of the form

$$L = a_0 (R/a_0)^{D}$$

where a_0 is the value of the scale used; R – distance in a straight line; D – fractal dimension. Near the critical point, the system can be considered as fractal self-similar, on scales $a_0 << r << \xi$, and as homogeneous on large scales. You can enter density ρ_N type

$$\rho_N = \frac{N}{r^d} = \begin{cases} r^{D-d}, & r < \xi \\ const, & r > \xi \end{cases} \quad (11)$$

The value ξ is the scale at which the density becomes constant (in NR this is the region in which nuclear fissions do not occur, but there is an uniform flux of neutrons escaping from the region with fissions, e.g., a reflector). Any intensive quantity behaves in the same way. Fractal behavior significantly affects the operation of nuclear reactors in the critical region. Thus, fractal characteristics determine the speed of propagation of disturbances in the supercritical regime, at $p > pc$. The chemical distance Rc between nodes i and j is defined as the minimum number of steps in which one can get from i to j, passing only through accessible nodes.

Having defined the ball Bc as a set of nodes for which $Rc \leq n$, we can define the chemical dimension as an indicator Dc such that the number of NB nodes belonging to Bc increases with n, as

$$NB \sim n^{Dc} \quad (12)$$

The value $Dc = D/DR$ is the ratio of two dimensions - the cluster dimension D and the dimension DR of the curve, the length of which gives us the chemical distance.

Another name for this dimension is the connectivity dimension. The effective speed of propagation of the disturbance front at $p > pc$, at local supercriticality is equal to $v \sim \tau^{(D_R-1)\nu}$; the indicator $\psi = (DR - 1)\nu$ effective speed determines the magnitude of the critical deceleration as p approaches pc from above. In classical theory $\psi = 0.5$, and $\nu \sim \tau^{0.5}$. This characteristic is important for assessing various types of time characteristics of a critical reactor and, ultimately, for the safety of nuclear power plants. If local supercriticality occurs in a nuclear reactor, then the speed of its propagation must be assessed based on fractal patterns. Many important properties of the kinetics of processes in nuclear radiation can be understood in more detail by considering the dynamics of processes on the Bethe lattice (taking into account delayed neutron sublattices); this applies to both lattice diffusion and other dynamic phenomena. From the fact that ψ=0.5, we find: $DR = 2$, $Dc = 2$ and $NB \sim n^2$.

For the Bethe lattice and Cayley trees $d = 6$, $D = 4$. Classical values of the critical exponents are valid for $d \geq 4$ [9]. Note that fractals can have an entire fractal dimension greater than the dimension of the space in which they exist; this is true, for example, for infinite clusters. Cayley trees are constructed in ultrametric hierarchical space. For a three-dimensional Euclidean embedding space in a cubic lattice, the value of the fractal dimension corresponds to the branching measure. This paper examines approaches to describing fission processes in nuclear reactors associated with percolation laws, with fractal dimension $D = 4$ and Laplacian fractals with $D = 2.4$. It is also possible to use other models, for example random trees [18].

Eq. (4) makes it possible to evaluate the influence of control parameters (using control rods, boric acid concentration, etc.) at the most critical point, with reactivity ρ equal to zero. The classical value in the theory of a self-consistent field of the critical index is δ=2. From Eq. (6), differentiating it with respect to $\tau \sim \rho$, we obtain: $\frac{dN}{d\rho} \sim -|p - p_c|^{-2}$, and the power coefficient of reactivity near the critical point is equal to $\alpha_N \sim \frac{d\rho}{dN} \sim -|p - p_c|^2 \sim -|k - 1|^2$.

An expression is also written for the behavior of the correlation function $g(r)$ at large r, when $g(r) \sim r^{-d+2-\eta}$, the classical value of the index $\eta = 0$, and for $d = 6$, $D = 4$:

$$g(r) \sim r^{-d+2} \sim r^{-4} \sim l^{-2}$$

Note that in the Cayley tree model, relations of the form (10) involve not the Euclidean distance r, but the chemical distance l between nodes. Thus, the chemical distance between the central node and an arbitrary node belonging to the l-th shell is equal to l [21]. The last relation l^{-2} corresponds to the dependence $l^{(d-D}$. The size of the critical cluster is determined by the relation $s_c \sim |\tau|^{-\Delta}$, where the value $\Delta = 2$.

For d=3, in a cubic lattice, $g(r) \sim r^{-d+2} \sim r^{-1}$. The total number of particles in the system N is related to the chemical distance in the system l by the relation $N \sim l^{Dc}$. The fraction of clusters with s nodes (the fraction of fission chains of s neutrons) in accordance with Eq.(9) and with the classical value of the index $\Delta = 2$ is equal to $n_s(\tau) = f(s)\exp(-As|\tau|^2)$, $n_s(p_s) = s^{-5/2}$.

The last expression shows that at the most critical point the fraction of fission chains of s neutrons is related to the value of s not by an exponential, but by a power-law dependence.

Percolation on Bethe lattices and the neutron multiplication factor

Although processes on Bethe lattices are considered, as a rule, for the case of an infinite lattice, in this work we consider the case of a finite lattice, which corresponds to a finite number of neutrons in the reactor. Taking into account a finite number of neutrons is necessary, for example, during reactor startup, in critical assemblies, in small reactors, etc. The results obtained can be useful for fast neutron reactors and for transient processes. In addition to the probability of percolation and the percolation threshold, there are a large number of other characteristics of the percolation process [1].

The percolation probability is interpreted as the probability of a self-sustaining chain reaction and the percolation threshold proportional to the neutron multiplication factor. A recurrence relation was obtained in [22] for the probability of percolation from a root vertex - the probability that a connected component of a configuration containing a root vertex (some starting point of the appearance of the first neutron in the system, giving rise to a chain reaction) reaches the opposite edges of the system. Mathematically, the size of the system and the connected component tends to infinity (conditionally), although real systems are finite. In [22], the value $P(n,c)$ denotes the probability of percolation from the root vertex to a distance n, $c=p=\lambda_f/(\lambda_f+\lambda_c)$. The value n in our problem is interpreted as the number of generations of neutrons in a chain reaction. The percolation threshold in [22] is the number $c_{c\infty}=\inf\{c: P(c)>0\}$. In [2], this quantity is called the critical probability of the appearance of a cluster extending over the entire lattice. Here $P(c) = \lim n\to\infty P(n,c)$, as in Eq.(2). The figure taken from [1] shows the behavior of the function $P(c=p)$. We will assume that the percolation threshold also exists for finite values of n,

$$c_{cn} = \inf\{cn : P(n,c) > 0\} \qquad (13)$$

The probability of percolation in the general case can behave as shown in Fig. 4 [1]. An important tool for studying percolation is the recurrence relation for the probability of percolation.

In [22], this relation is written in the form

$$P(n+1,c)=c[1-(1-P(n,c))^s];\ P(0,c)=c \qquad (14)$$

where $s=\overline{\nu}$. In the continuous approximation there is a derivative of $P(n,c)$ with respect to c. To study the behavior of $P(n,c)$, for example, series expansion or determination of the inflection point, one must also know the second derivative. These probabilities can be given a physical meaning. The expressions for derivatives obtained from Eq.(14) have the form

$$f(n,c)=dP(n,c)/dc;$$
$$f(n+1,c)=1-(1-P(n,c))^{s-1}[1-P(n,c)-csf(n,c)]; \quad (15)$$
$$f(0,c)=1$$

$$r(n+1,c)=s(1-P(n,c))^{s-2}[2(1-P(n,c))(dP(n,c)/dc)-$$
$$-c(s-1)\,(dP(n,c)/dc)^2+c(1-P(n,c))r(n,c)];$$
$$r(n,c)=d^2P(n,c)/dc^2; \quad (16)$$
$$r(0,c)=0.$$

From Eq. (14) we obtain a picture on behavior of percolation probability for the Bethe lattice (Fig. 3), which differs from that shown in Fig. 2, obtained for simple lattices. Probability $P(n,c)$ in Fig. 3 is calculated at n=750. The vertical line shows the value $cc=\overline{\nu}^{-1}$ at $n\to\infty$.

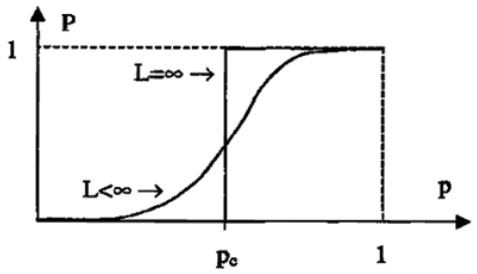


Fig. 2. Probability of percolation occurrence P depending on the fraction of filled nodes $p=c$ (smooth curve corresponds to a lattice of finite size, stepped curve to an infinitely large lattice)

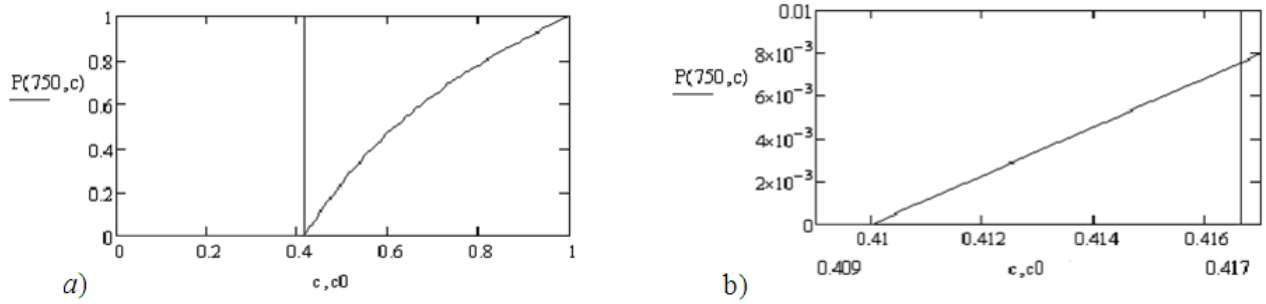


Fig. 3. Probability of percolation for the Bethe lattice, n=750 (ranges of variation with - 0…1 (*a*) and 0.409…0.417 (b))

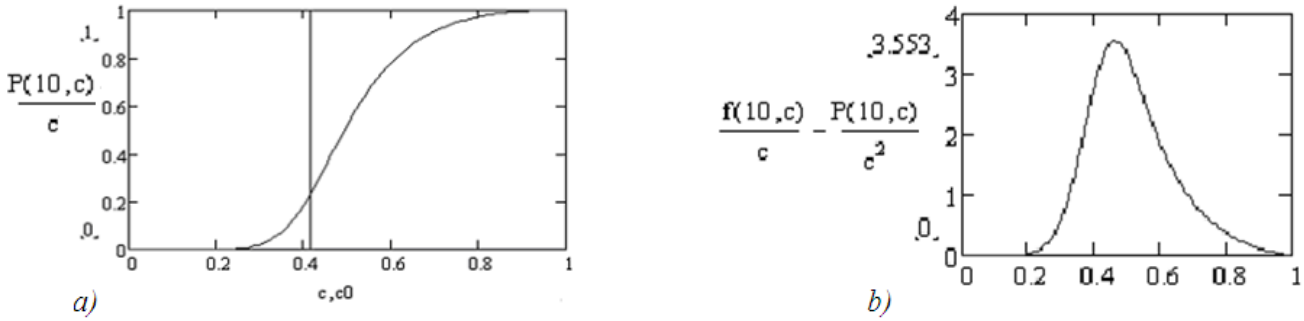


Fig. 4. Behavior of the conditional probability $P(10,c)/c$ (a) and the derivative with respect to c of the function $P(10,c)/c$ (b)

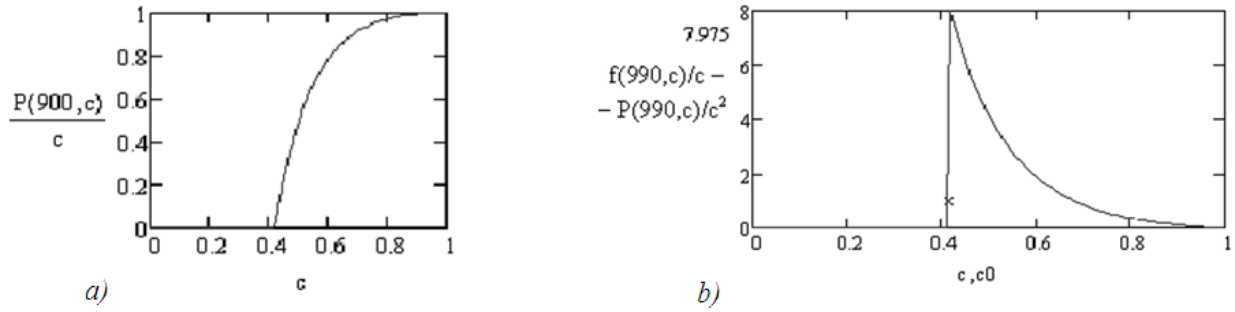


Fig. 5. Behavior of the function $P(900,c)/c$ (a) and the derivative with respect to c of $P(990,c)/c$ (b)

Since the system is finite, the number of generations is n=750, the critical probability is not equal to $c_c=\overline{\nu}^{-1}$. This can be seen in Fig. 3b, which shows a different range of changes in the value of c - from 0.409 to 0.417, and not from 0 to 1, as in Fig. 3a. From Fig. 3b it is clear that the critical probability value for n=750 is less than for an infinite lattice, when $c_c=\overline{\nu}^{-1}$, $c_{c750}\approx 0{,}41<\overline{\nu}^{-1}$. In the interval from c=0.41 to $c_{c\infty}$ there is a non-zero probability of about $10^{-3}\ldots 8\cdot 10^{-3}$ of percolation or, for reactors, the probability of a self-sustaining chain reaction of fission of uranium nuclei. Thus, for finite systems the percolation threshold and the multiplication factor are less than unity.

Fig. 2 differs from Fig. 3. Fig. 2 (a similar figure, showing also the derivative $dP(n,c)/dc$, is in [23]) shows the conditional probability $P(n,c)/c$, corresponding in our interpretation to the fact that the initial neutron enters the fissile nucleus. Fig. 4 shows the behavior of the function $P(10,c)/c$ and the derivative with respect to c of this function, similar to Fig. 2 and the figure from [23]. But this kind of behavior is typical for small values of n. As n increases (Fig. 5), the picture is more reminiscent of Fig. 3 and Fig. 6 for the derivative with respect to c of $P(n,c)$.

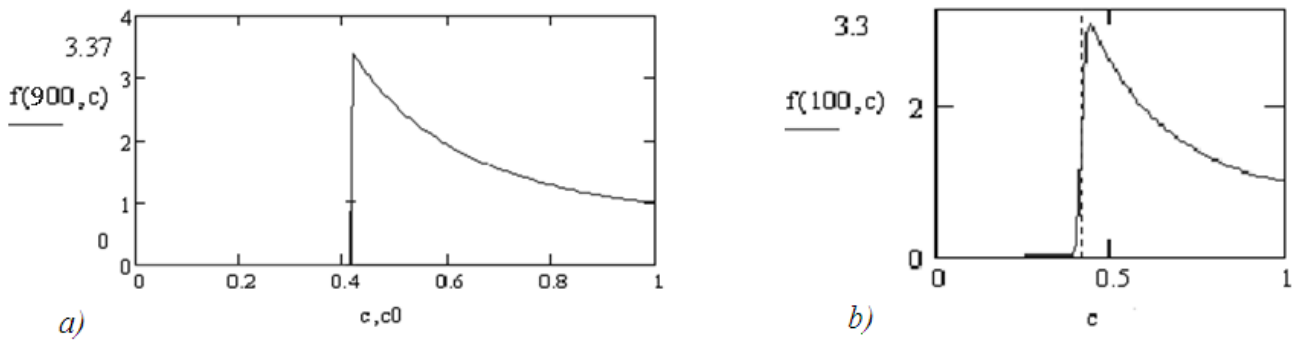


Fig. 6. Dependence of the derivative with respect to c on $P(n,c)$, n=900, *a*) and n=100, b)

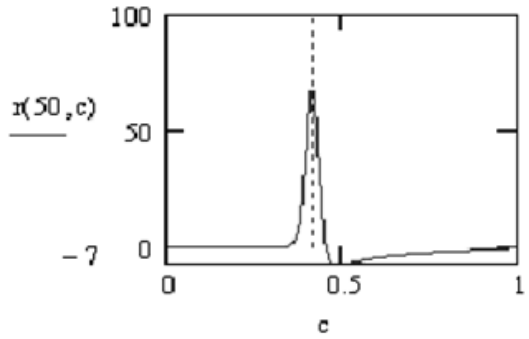


Fig. 7. Behavior of the second derivative with respect to c of $P(n,c)$, n=50

The behavior of functions (15) and (16) has the form shown in Figs. 6, 7. It can be seen that the form of these functions is also related to the position of the percolation threshold, shown by the vertical dashed line.

Fig. 8 shows the dependence of $P(n,c_0)$ on n for a fixed value $c=c_0=\overline{\nu}^{-1}=c_{c\infty}$ in the interval of changing n from 2,000 to 3,000. This function also behaves in other intervals of changing the number of neutron generations n. This dependence is approximated by the function $1.43/n$. For values of c other than $c=c_0$, this dependence does not hold.

In [23] the value of the critical region is described as $c \approx cc+B/L+...$, where B is a constant; L is the size of the system. It was noted in [24] that the Bethe lattice size is proportional to lnN. Since $N=n$, this corresponds to the expression $1.43/n$. It is possible to estimate the time during which the values of $c_0=c_{c\infty}$ reach a given level. For example, the value 10^{-6} is achieved in $n=1{,}43\cdot10^6$ generations of neutrons. For thermal neutron reactors, where the average lifetime of a generation taking into account delayed neutrons is 10^{-1} s, this time is $1.43\cdot10^5$ sec = 1.655 days. For fast neutron reactors, where the average lifetime of a neutron generation is $10^{-4}\ldots10^{-8}$ s, this time decreases by 3…7 orders of magnitude. The considered methods for determining the critical point can be useful during the startup of a reactor or during its transient processes, when the probability value c changes due to manipulations with absorber rods.

The rate of change of the function $P(n,c)$ depending on n can be described by its derivative with respect to n, more precisely, by its discrete analogue, the value $P(n+1,c_0)-P(n,c_0)$. Calculations show that this value, taken with a negative sign, for $c_0=cc\infty$ is well described by the dependence $1.43/n^2$. The second derivative, the function $P(n+1,c0) - 2P(n,c0) + P(n-1,c0)$, is described by the dependence $2.83/n^3$. The dependence of the k-th derivative with respect to n for $c_0=c_{c\infty}$ is proportional to $n-(k+1)$. The behavior of the discrete analogue of the dependence on c on the fourth derivative of $P(n,c)$ with respect to n is shown in Fig. 9. The behavior of the analogue of the second derivative on a smaller scale of change in the value of c is shown in Fig. 10.

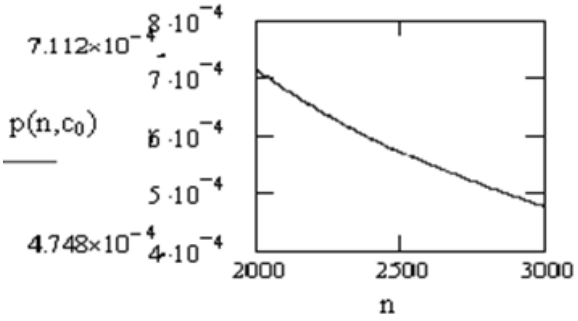


Fig. 8. Dependence of $P(n,c_0)$ on n (2000<n<3000) at a fixed value $c=c_0=\overline{\nu}^{-1}=c_{c\infty}$

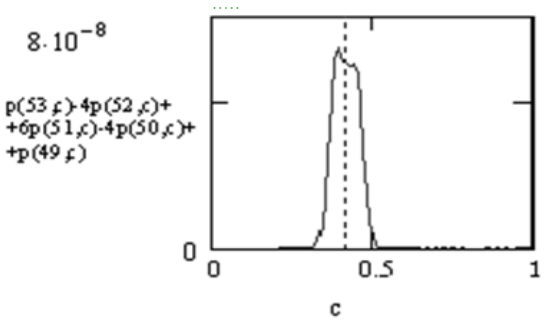


Fig. 9. Behavior of the discrete analogue of the dependence on c on the fourth derivative with respect to n on $P(n,c)$

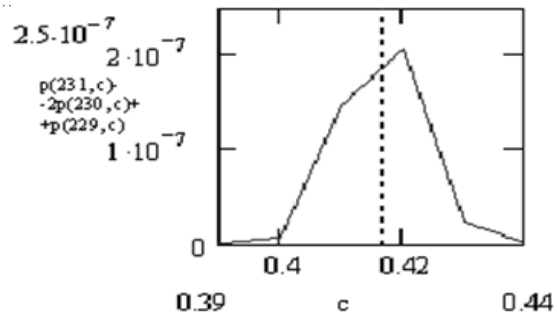


Fig. 10. Dependence on c of the analogue of the second derivative with respect to n of $P(n,c)$ (0.39<c<0.44)

The dependence of the difference between the second derivatives at c=0.42 and c=0.4 points on *n* is shown in Fig. 11. Similar dependencies are written for derivatives according to Eq. (5), (6). Thus, behavior of the discrete analogue of the first derivative with respect to *n* from the second derivative of *P(n,c)* with respect to *c* is shown in Fig.12; the dependence of this quantity on *c* for fixed *n*=50 and *n*=15 is shown in Fig. 13a and 13b, respectively.

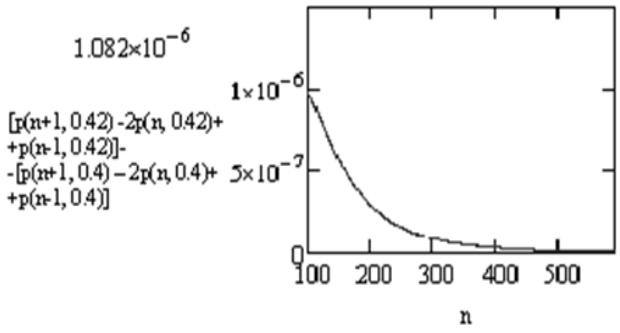


Fig. 11. Dependence on *n* of the difference between the second derivatives with respect to *n* of *P(n,c)* at points *c*=0.42 and *c*=0.4

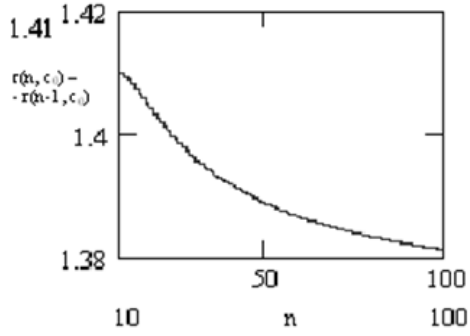


Fig. 12. Dependence on *n* of the discrete analogue of the first derivative with respect to *n* on the second derivative of *P(n,c)* with respect to *c*

The dependence on *n* of the third derivative with respect to *n* of *r(n,c)*, the second derivative with respect to *c* of *P(n,c)* is shown in Fig. 14. Position of maxima and minima in Fig. 14 may be related to the boundaries of the critical region. Fig. 13 shows that the negative peaks for the first derivative with respect to *n* become closer together as *n* increases.

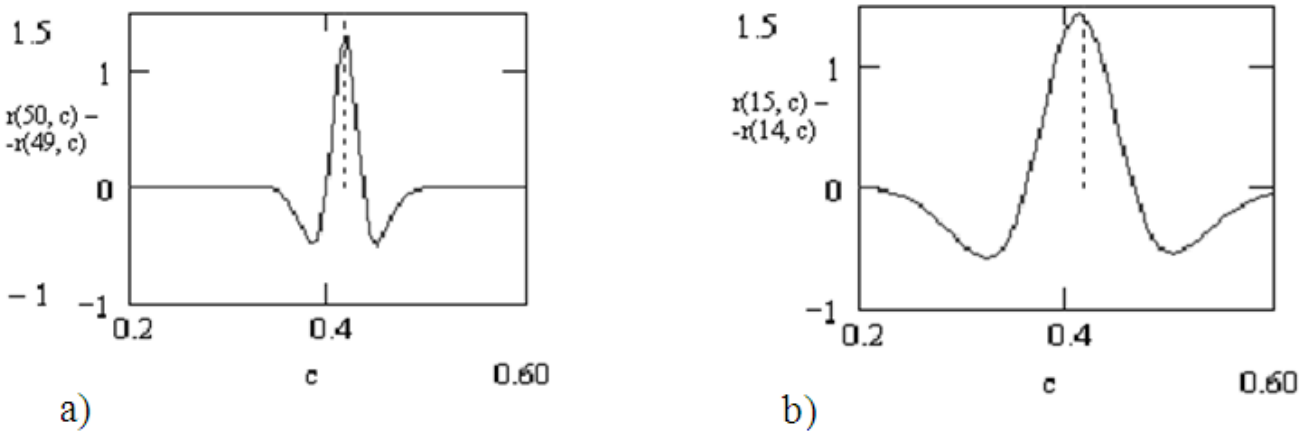


Fig. 13. Behavior of the dependence on *c* of the discrete analogue of the first derivative with respect to *n* on the second derivative of *P(n,c)* with respect to *c* for fixed *n*=50 (a) and *n*=15 (b)

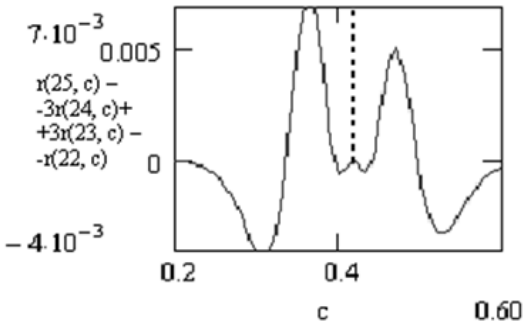


Fig. 14. Dependence of the third derivative with respect to *n* on the second derivative of *P(n,c)* with respect to *c* (6) for *n*= 25

Models of irreversible growth. Dimensions

The fractal dimension of tree-like structures *D* in a reactor with thermal neutrons and uranium as fuel (for example, VVER) is equal to the average number of secondary neutrons in one fission event, *D* = 2.4. This follows from [5, Ch. 16], where it is noted that for trees with infinitely thin trunks, the fractal dimension *D* serves as a measure of branching. However, in this work we also used the Cayley tree model, for which the fractal dimension is *D* = 4. The difference in dimensions is caused by different embedding spaces. Another significant reason for the difference in fractal dimensions is the multifractality of the processes of fission and neutron transfer in nuclear reactors.

In [25, 26], fractal structures simulating electrical breakdown are considered. The same kind of phenomena is observed in the processes of crack formation. The equations describing these processes are compared with the diffusion model of neutron transfer in the reactor. The study is carried out on a cubic lattice in a three-dimensional embedding space. The fractal dimension *D* for this case is close to the branching measure of 2.4 for uranium nuclei in the reactor. The human circulatory system has almost the same dimension: 2.4 – 2.6.

The average density of particles in a cluster behaves as it moves away from the center, as in relation (11). It can be expressed by the formula:

$$\rho_N = \rho_{N0}(\frac{r_0}{r})^{\alpha_1}\Phi(\frac{R-r}{\sqrt{2}\Delta r}) \quad (17)$$

where $\alpha_1 = d - D$, d– dimension of space, $\alpha_1 = d - D = 0.6$ for $d = 3$, $D = 2.4$; Δr is the depth of penetration of a particle into a cluster of size R; r - distance from the cluster center; $R >> R - r_0 >> \Delta r$; $\Phi(x)$- probability integral; r_0 is a certain minimum radius (such as the mean free path of a neutron). For the total number of particles in the cluster we find (for $\Delta r << R$)

$$N = \int_0^R 4\pi r^2 \rho(r)dr = 4\pi r_0^{\alpha_1}\rho_0 \frac{R^D}{D}[1-(\frac{2}{\pi})^{1/2} D\frac{\Delta r}{R}], \quad D = 2.4 \quad (18)$$

It is possible to estimate the dependence of the particle penetration depth Δr when connecting to a cluster from the task parameters. The expressions obtained in [25] for the average distance between nearest branches may also be of some interest in reactor theory

$$\langle \lambda(r) \rangle = r^{(\frac{d-D}{d-1})} \quad (19)$$

and other parameters. If $\gamma(r)$ is the number of lines of neutron trajectories of one chain intersecting a circle of a given radius; $\gamma \pm (r)$ is the radial density of branching points (fissile nuclei) (+) and free ends (-) (neutron absorption points), and $L(r)$ is the total length of the neutron trajectory lines of one chain in a circle of radius r, then the following are satisfied similarity relations [26]

$$L(r) \sim r^D; \quad \gamma(r) \sim r^{D-1}; \quad \gamma_{\pm}(r) \sim r^{D-2} \quad (20)$$

In addition to those listed, you can specify other fractal characteristics that can prove to be effective for solving various problems of operation and studying the behavior of nuclear reactors. Relations (19) and (20) are written for $d = 3$, $D = 2.4$ in the form

$$\langle \lambda(r) \rangle \sim r^{0.3}; \quad L(r) \sim r^{2.4}; \quad \gamma(r) \sim r^{1.4}; \quad \gamma_{\pm}(r) \sim r^{0.4}$$

Such an internal indicator of a cluster is used in [18] as the internal dimension of its skeleton d_l^S. The cluster skeleton is defined as the set of all shortest paths connecting the cluster nodes with the surrounding shell, located at a chemical distance $L << l$ from the region under consideration. The skeletons of percolation clusters in a space of any number of dimensions at a critical concentration are linear in the sense of the chemical distance, i.e. $d_l^S = 1$. General relationships between dynamic indicators describing diffusion and statistical indicators describing the structure of the tree are obtained:

$$d_w^l = 2 + D_c - d_l^S; \; d_w = D(2 + D_c - d_l^S)/D_c; \; \bar{d} = 2D_c/(2 + D_c - d_l^S),$$

where dw is the diffusion index; d_w^l – chemical diffusion indicator; $d = 2D/dw$ is the fracton (or spectral) dimension characterizing the connectivity of an object, the density of states and its spectral properties [18]. When $d_l^S = 1$,

$$d_w^l = 1 + D_c; \; d_w = D(1 + D_c); \; \bar{d} = 2D_c/(1 + D_c).$$

For Cayley trees $Dc = 2$, $dw = 6$, $d_w^l = 3$, $\bar{d} = 4/3$. These are not all the results of fractal and percolation approaches to describing the internal and dynamic properties of neutron clusters that form tree-like structures of fission chains in nuclear reactors.

Thus, in [1, 2] the dimension of the shell, the boundary of a connected cluster, is defined as $Dh = (1 + \nu)/\nu$, equal to 3 for percolation on Cayley trees. In [27], the dependence of the cluster perimeter P on the number of nodes in it s is given by the relation

$$P = [s(1 - pc)/pc] + As^{\sigma},$$

where A is a constant, $\sigma = 1/\nu D$, equal to 1/2 for percolation on Cayley trees. In [27], the fractal dimension of the cluster perimeter dG (equal to 2 for percolation on Cayley trees), obtained in the butterfly walk (cluster growth) model, was considered, and a new relation was proposed for the chemical dimension $Dc = D/(D - dG)$.

Conclusions

The importance of the relations of the percolation theory for neutron processes in a reactor is already evident from the fact that they allow us to immediately obtain the neutron multiplication equation and the equation for the critical size of the reactor (Eqs.(6) and (7), interpreting the general relations of the percolation theory Eqs.(2) and (3)).

Eqs. (6) and (7) obtained directly from the relations of percolation theory indicate effectiveness of this approach in the theory of neutron processes in a reactor. The relation for the speed of propagation of a disturbance at local supercriticality should prove useful. Many other expressions of percolation theory applied to reactors may be also of interest.

This is apparently due to the fact that percolation is a critical process that presupposes the existence of a threshold, a critical point. At the threshold, flow occurs along a fractal set, the geometry of which is determined by criticality. The geometric characteristics of a fractal are independent of the microscopic properties of the medium. Below the critical point, kinetic processes are limited to a finite region of phase space, scattering, absorption and other neutron processes. At the critical point, the fractal set, formed when the free energy of the statistical ensemble decreases, becomes decisive. The behavior of the system under slow forcing influences on it tends to self-organized criticality - a singular non-equilibrium (quasi)stationary state [28, 29]. Stationary non-equilibrium states on fractal structures are chaotic, turbulent in nature. In [21, 30], the Lorentz model is used to study them.

The kinetics and processes of transfer in fractal reactor structures require a separate detailed study [4, 11, 12, 21]. In the region of the critical point, long-range correlation effects appear, manifested in the non-Gaussian behavior of kinetic processes, determined by the topological invariants of self-similar fractal sets. Transfer processes at the percolation threshold are discussed in [11, 12]. Fractional derivative equations are used that take into account the effects of memory, non-locality and intermittency.

A deeper analysis of the hierarchical structure of neutron trajectories in nuclear reactors, based on the results of [21], shows that the behavior of the probabilities of a chain reaction occurrence is determined by the probabilities $c = p = \lambda_f/(\lambda_f + \lambda_c)$, the degree of criticality, and proximity to the critical point.

Depending on this proximity, three (more precisely, four, if we highlight the critical point itself) main modes of behavior are distinguished: subcritical and supercritical (in them the laws of behavior (4) and (10) differ only in sign), critical (11), (14), and critical point (12), (13). In the traditional theory of nuclear reactors, only subcritical and supercritical regimes and the critical point are studied, although in the general theory of phase transitions a critical region is necessarily present. This is due to the fact that neutrons do not interact; the values of the classical critical indices are valid for them (as for a self-consistent field) [31]. In the stationary operating state of reactors there are many neutrons, their number can be considered infinitely large. In this case, the critical region contracts to a critical point. Note that in this case the equation for P in the continuum limit can be solved exactly. But the integrals are complex, and it is difficult to express the function P explicitly.

The critical region itself, as shown in [32], has a complex three-member structure. In [32], three modes of critical behavior of nuclear reactors were discovered, depending on the sign of control actions and feedbacks, the boundaries of these modes were found, and it was shown that in the region of the critical point the time behavior is power-law. Time is proportional to the number of generations, and this behavior is characteristic of self-similar irregular trees [21]. At the most critical point, the total number of neutrons is proportional to time, which corresponds to a degenerate tree. Thus, neutron trajectories vary depending on the probability c and the multiplication factor. In the subcritical (and supercritical) region the movement occurs along regular trees, in the critical region along self-similar irregular trees, at the critical point along a degenerate tree. Above the critical point, but in the critical region - again using self-similar irregular trees. In the supercritical region the movement occurs along regular trees.

References

[1] Tarasevich, Y.Y. (2002) *"Percolation theory, applications, algorithms"*, Moskow, URSS (in Russian)

[2] Feder, E. (1988) *"Fractals"*, Plenum Press, New York, US

[3] Bozhokin, S.V., Parshin, D.A. (2001) "Fractals and multifractals", *I*zhevsk: NITs "*Regular and Chaotic Dynamics*", 128 p. (in Russian)

[4] Olemskoi A.I., Flath A.Y. (1993) "Using fractal concepts in condensed-matter physics", *Uspehi fizicheskih nauk*., Vol. 163(12), p. 1-50 (in Russian)

[5] Mandelbrot, B.B. (1977) B.B. *"Fractals: Form, Chance, and Dimension",* 753 p., Ed. Freeman, San Francisco, USA
Mandebrot, B.B. (1982) *"The Fractal Geometry of Natura",* 530 p., Ed. Freeman, San Francisco, USA

[6] Ryazanov, V.V., Turbin, A.F. (1991) "Tree structures, the problem of percolation and fractal phenomena in the multiplying medium", Abstracts seminar "Fractal objects in mathematics, physics and biology", *Sloviansk*., Vol.14, p.17, Soc. "Knowledge" publ. house, Kiev, Ukraine (in Russian).

[7] Efros, A.L. (1982) *"Physics and geometry of disorder",* 176 p., Ed. Nauka, Moscow, Russia (in Russian)

[8] Patashinskii, A.Z., Pokrovskii, V.L. (1975) *"Fluctuation theory of phase transitions",* 255 p., Ed. Nauka, Moscow, Russia (in Russian)

[9] Ma, Sh. (1980) *"Modern Theory of Critical Phenomena",* 374 p., Ed. Wiley, New York, USA

[10] de Gennes, P. (1982) *"Scaling ideas in polymer physics"*, 368 p., Ed. Wiley, New York, USA

[11] Zelenyi, L.M., Milovanov, A.B. (2004) "Fractal topology and strange kinetics: from percolation theory to problems in cosmic electrodynamics", *Uspehi fizicheskih nauk*., Vol. 174(8), pp. 809 - 852 (in Russian)

[12] Uchaikin, V.V. (2003) "Similar anomalous diffusion and stable laws", *Uspehi fizicheskih nauk*., Vol. 173(8), pp. 847 - 876 (in Russian)

[13] Nigmatullin, R. (1992) "Fractional integral and its physical interpretation", *Teoreticheskaya i matematicheskaya fizika*, Vol. 90(3), pp. 354 - 368 (in Russian)

[14] Zel'dovich, B., Sokolov, D.D. (1985) "Fractal dimension and related matters", *Uspehi fizicheskih nauk.*, Vol. 146(3), pp. 493 - 517 (in Russian)

[15] Smirnov, B.M. (1986) "Fractal clusters", *Uspehi fizicheskih nauk.*, Vol. 149(2), pp. 177 - 210 (in Russian)

[16] Sokolov, I.M. (1986) "Dimensions and other geometric critical exponents in percolation theory", *Uspehi fizicheskih nauk*., Vol. 150(2), pp. 229 - 265. (in Russian)

[17] Harari, F., Palmer, E. (1977) "Enumeration of graphs", 324 p., Ed. Wiley, New York, USA

[18] Havlin, S. (1986) "Statistical and dynamical properties of fractal aggregates that do not contain loops", *Fractals in Physics*, pp. 498-506, Ed. Wiley, New York, USA

[19] Turbin, A.F., Pratsevity, N.V. (1992) *"Fractal sets, functions, distribution"*,207 p, Ed Naukova Dumka, Kiev, Ukraine (in Russian)

[20] Harris, T. (1966) *"Theory of branching random processes"*, 355 p., Ed. Wiley, New York, USA

[21] Olemskoi, A.I. (2009) *"Synergetics of complex systems. Phenomenology and statistical theory"*, 379 p., Ed. Krasand, Moscow (in Russian)

[22] Antonova, E.S., Virchenko, Y.P. (2010) "Continuity of the probability of percolation of Bernoulli random fields on homogeneous tree graphs", *Nauch. statements of Belgorod State Univ.*. Series: Math. Phys., Vol. 23(94), No. 21, pp. 11-21 (in Russian)

[23] Stauffer, D. (1994) *"Introduction to percolation theory"*, 2nd rev., 179 p., editors D. Stauffer, A. Aharony, Ed. Taylor & Francis, London, UK

[24] Cohen, R. (2010) *"Complex networks: Structure, Robustness and Function"*, 330 p., editors R. Cohen, S. Havlin, Ed. University Press, Cambridge, UK

[25] Pietronero, L., Everts, K., Wiesmann, H. (1986) "Properties similarity growing area and the capacity of the Laplacian fractal", *Fractals in Physics*, pp.221-226, Ed. Wiley, New York, USA

[26] Wiesmann, H., Pietronero, L. (1986) "Properties of Laplace fractals in the breakdown of dielectrics in two and three dimensions", *Fractals in Physics*, pp. 210-20, Ed. Wiley, New York, USA

[27] Margolin, A. (1986) "The fractal dimension of the perimeter growth", *Fractals in Physics*, pp.507-512, Ed. Wiley, New York, USA

[28] Bak, P., Tang, C., Wiesenfeld, K. (1988) "Self-organized criticality", *Phys. Rev. A*, Vol. 38(1), pp. 364 -374

[29] Schroeder, M. (1992) *"Fractals, Chaos, Power Laws. Minutes from an Infinite Paradise"*, 528 p., Dover Publications

[30] Shuda, I.A. (2011) *"Influence of the hierarchical structure and self-similarity to self-organization of complex systems"*, Diss. Ph.D., Sumy (in Russian)

[31] Ryazanov, V.V. (2024) "Relationship between chain reactions in a nuclear reactor and multifractality", http://arxiv.org/abs/2402.18972

[32] Boiko, R.V., Ryazanov, V.V. (2002) "Stochastic Model of a Nuclear Power Reactor", *Atomic energy,* Vol. 93(2), pp. 625 – 634.